

NEW QUANTUM GROUPS BY DOUBLE-BOSONISATION

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Abstract We obtain new family of quasitriangular Hopf algebras $\mathbb{C}_q^{0|n} \bowtie U_q(\widetilde{su_n}) \bowtie \mathbb{C}_q^{0|n}$ via the author's recent double-bosonisation construction for new quantum groups. They are versions of $U_q(su_{n+1})$ with a fermionic rather than bosonic quantum plane of roots adjoined to $U_q(su_n)$. We give the $n = 2$ case in detail. We also consider the anyonic-double of an anyonic $(\mathbb{Z}/n\mathbb{Z}$ -graded) braided group and the double-bosonisation of the free braided group in n variables.

Keywords: braided group – bosonisation – quantum group – quantum double – triangular decomposition – non-standard – anyonic – super – C-statistical

1 Introduction

There are two well-known general constructions for quasitriangular Hopf algebras or ‘strict quantum groups’ (with universal R-matrix), namely (i) Drinfeld’s quantum double[1] of any Hopf algebra and (ii) the R-matrix approach introduced for general R-matrices in [2], where the existence of a universal R-matrix functional on $A(R)$ was established for the first time (this is not a result to be found in the standard FRT work[3]; it is needed when the universal R-matrix is not known by other means). Recently, in [4], we introduced a third and more powerful general construction for quasitriangular Hopf algebras called *double-bosonisation*. It associates to any braided group B covariant under a background quantum group H a quasitriangular Hopf algebra $B^{\star\text{op}} \bowtie H \bowtie B$ built on $B^{\star} \otimes H \otimes B$ and consisting of H extended by B as additional ‘positive roots’ and its dual B^{\star} as additional ‘negative roots’. The construction is more powerful than the quantum double because one can reach directly to the usual $U_q(g)$ (the quantum double is too big and one has to quotient it). As an application, we can now build up $U_q(g)$ by induction, adjoining to $U_q(su_2)$ a quantum-braided plane of roots to get to $U_q(su_3)$ etc. See [4]. One can also go up the $U_q(so_n)$ series etc., for example constructing the conformal algebra $U_q(so_6)$ from $U_q(so_4)$ by adjoining a q-Euclidean braided plane of roots[5].

Moreover, because each double-bosonisation has a decomposition into its three tensor factors, we obtain an ‘inductive block-basis’ for each family of the classical families of quantum groups. For the A_n series it is:

$$\dots \mathbb{C}_q^{n\star\text{op}} \bowtie (\mathbb{C}_q^{(n-1)\star\text{op}} \bowtie \dots \bowtie (\mathbb{C}_q^{\star\text{op}} \bowtie U(1) \bowtie \mathbb{C}_q) U(1) \bowtie \dots U(1) \bowtie \mathbb{C}_q^{n-1}) U(1) \bowtie \mathbb{C}_q^n \dots \quad (1)$$

where we can go up to arbitrary n . This shows $U_q(su_{n+1})$ built up by starting with $U(1)$, adjoining a braided line \mathbb{C}_q to obtain $U_q(su_2) = \mathbb{C}_q^{\star\text{op}} \bowtie U(1) \bowtie \mathbb{C}_q$, then adjoining a braided plane \mathbb{C}_q^2 to obtain $U_q(su_3)$ from

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this, etc. Every element of $U_q(su_{n+1})$ decomposes into the product of a unique element of the tensor product of these blocks. Also, wherever we take the brackets (on either side) we obtain a Hopf algebra. For example,

$$(\mathbb{C}_q^{(n-1)*\text{op}} \bowtie \dots \bowtie (\mathbb{C}_q^{*\text{op}} \bowtie U(1) \bowtie \mathbb{C}_q) U(1) \bowtie \dots U(1) \bowtie \mathbb{C}_q^{n-1}) U(1) \bowtie \mathbb{C}_q^n = U_q(\widetilde{su_n}) \bowtie \mathbb{C}_q^n \quad (2)$$

is a sub-Hopf algebra of $U_q(su_{n+1})$, its natural ‘maximal parabolic’ inhomogeneous quantum group. Thus we obtain not only the classical q -deformations but a natural description of how they are built up from each other. Thus, double-bosonisation is the quantum analogue of adjoining a node to a Dynkin diagram and thereby extending its Lie algebra to a larger one. Moreover, fixing bases of each of the \mathbb{C}_q^n , we obtain a concrete *inductive basis* for the entire A -series quantum groups. Similarly for the other series of $U_q(g)$.

On the other hand, it is clear that this construction can be used just as well to obtain non-standard quantum groups. Basically, the double-bosonisation construction generates not a line but a *tree of quantum groups*: At each node of the tree we have the choice to adjoin any braided group covariant under the quantum group at that node. Some of the nodes in the tree will be classical in the sense that they have smooth $q \rightarrow 1$ limits and some others will be purely quantum with no limit. For example, at the node $U_q(su_2)$ we can choose \mathbb{C}_q^2 giving $U_q(su_3)$ as above, \mathbb{R}_q^3 giving $U_q(so_5)$, $\mathbb{C}_q^{0|2}$ giving a non-standard quantum group without a classical limit, and probably other choices as well. From each of these nodes we have still more choices. The enumeration of all braided groups covariant under each the quantum group at each node leads to an enlarged ‘Dynkin diagram system’ describing the tree generated by all possible double-bosonisations.

In Section 3 we demonstrate this theory with an explicit computation of the non-standard quantum group obtained from the node $U_q(su_2)$ by adjoining $\mathbb{C}_q^{0|2}$. There is obviously an infinite number of new quantum groups to be obtained this way, and a combinatorial challenge to elaborate the full tree structure. In particular, at each $U_q(su_n)$ node we have a fermionic partner $\mathbb{C}_q^{0|n}$ to the bosonic quantum plane \mathbb{C}_q^n , giving a fermionic cousin of $U_q(su_{n+1})$. Moreover, being an abstract construction, double bosonisation is not tied to any q at all, and works as well for discrete quantum groups or at roots of unity. We give some examples of this type in Section 4.

The paper begins in Section 2, where we recall the abstract bosonisation construction from [4]. Full proofs are in [4], i.e. here we announce the main results only. In fact, [4] was rejected after several months with Journal of Algebra on rather trivial grounds, requiring resubmission elsewhere.

2 Abstract Double-Bosonisation Construction

This preliminaries section announces/recalls the abstract double-bosonisation construction from [4], as needed for the construction of the two examples in later sections. The reader can also read the latter first and keep this section for reference as a brief introduction to [4].

We assume in this section that the reader is familiar with the basic ideas of quantum groups and braided groups. We use the notations from the textbook [6]. Recall only that a quantum group (H, \mathcal{R}) generates a braiding $\Psi : B \otimes B \rightarrow B \otimes B$, and a braided group B means an H -covariant algebra and coalgebra which are compatible in the braided sense[7] that the coproduct $\underline{\Delta} : B \rightarrow B \underline{\otimes} B$ is an algebra homomorphism. Here $B \underline{\otimes} B$ has the braided tensor product algebra structure

$$(b \otimes d)(c \otimes e) = b\Psi(d \otimes c)e, \quad \Psi(d \otimes c) = c\triangleleft \mathcal{R}^{(1)} \otimes \triangleleft \mathcal{R}^{(2)}, \quad \forall b, c, d, e \in B. \quad (3)$$

Let B^* denote a braided group dual to B in the braided sense of an evaluation pairing $\text{ev} : B^* \otimes B \rightarrow \mathbb{C}$ being given[7]. This section works over a general field in place of \mathbb{C} here, or (with suitable care) over a commutative ring such as $\mathbb{C}[[\hbar]]$. We take conventions with all quantum group actions \triangleleft from the right. The main formulae with left actions \triangleright are summarized in the appendix of [5].

Theorem 2.1 [4] *There is a unique Hopf algebra structure $B^{\star\text{op}} \bowtie H \ltimes B$ on $B^\star \otimes H \otimes B$, the double-bosonisation, containing $B^{\star\text{op}}, H, B$ as subalgebras, with the cross relations and coproduct*

$$\begin{aligned} bh &= h_{(1)}(b \triangleleft h_{(2)}), \quad ch = h_{(2)}(c \triangleleft h_{(1)}), \quad \underline{b}_{(1)} \mathcal{R}^{(1)} \underline{c}_{(1)} \text{ev}(\underline{c}_{(2)} \triangleleft \mathcal{R}^{(2)}, \underline{b}_{(2)}) = \text{ev}(\underline{c}_{(1)}, \underline{b}_{(1)} \triangleleft \mathcal{R}^{(1)}) \underline{c}_{(2)} \mathcal{R}^{(2)} \underline{b}_{(2)} \\ \Delta b &= \underline{b}_{(1)} \triangleleft \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \underline{b}_{(2)}, \quad \Delta c = \mathcal{R}^{(1)} \underline{c}_{(1)} \otimes \underline{c}_{(2)} \triangleleft \mathcal{R}^{(2)} \end{aligned}$$

for all $b \in B$, $c \in B^\star$, $h \in H$.

The paper [4] provides several other conventions in which this can be presented (in particular, the above is *not* the most natural from the point of view of a left-right symmetry interchanging the roles of B^\star, B). It also proves several results about the double-bosonisation. The first is that H, B and $B^{\star\text{op}}, H$ generate sub-Hopf algebras $H \ltimes B$ and $B^{\star\text{op}} \bowtie H$ respectively. These are usual *bosonisations* associated to any braided group by the author's earlier bosonisation construction[8]. The second is that, being built on the triple tensor product, there is an explicit product law

$$\begin{aligned} (c \otimes h \otimes b) \cdot (d \otimes g \otimes a) &= (d_{(2)} \triangleleft \mathcal{R}_3^{- (1)} S h_{(1)}) c \otimes h_{(2)} \mathcal{R}_1^{(1)} \mathcal{R}_4^{- (2)} g_{(1)} \otimes (b_{(2)} \triangleleft \mathcal{R}_2^{(1)} g_{(2)}) a \\ &\quad \text{ev}(\underline{d}_{(1)} \triangleleft \mathcal{R}_4^{- (1)} \mathcal{R}_5^{- (1)}, \underline{b}_{(1)}) \text{ev}(\underline{g}_{(3)} \triangleleft \mathcal{R}_3^{- (2)} \mathcal{R}_1^{(2)} \mathcal{R}_5^{- (2)} \mathcal{R}_2^{(2)}, \underline{b}_{(3)}) \end{aligned} \quad (4)$$

between general elements $c, d \in B^\star$, $h, g \in H$ and $b, a \in B$. Here $\mathcal{R}_1, \dots, \mathcal{R}_5$ are five copies of the universal R-matrix of H . The third result is that when B has a basis $\{e_a\}$ with dual basis $\{f^a\}$ of B^\star (a strict duality) then the double-bosonisation has a universal R-matrix

$$\mathcal{R} = \exp_B^{-1} \cdot \mathcal{R}, \quad \exp_B^{-1} = f^a \otimes \underline{S} e_a. \quad (5)$$

The fourth is that the double-bosonisation acts covariantly on B itself. The (right) action is

$$v \triangleleft b = (\underline{S} \underline{b}_{(1)} \triangleleft \mathcal{R}^{(1)}) (v \triangleleft \mathcal{R}^{(2)} \underline{b}_{(2)}), \quad v \triangleleft c = \text{ev}(\underline{S}^{-1} c, v_{(2)} \triangleleft \mathcal{R}^{(2)}) v_{(1)} \triangleleft \mathcal{R}^{(1)} \quad (6)$$

for all $v, b \in B$ and $c \in B^\star$, and the given action \triangleleft of H on B .

The paper [4] also relates the double-bosonisation to the quantum double of the usual bosonisation. We adopt conventions in which $D(H) = H^{\star\text{op}} \bowtie H$. More precisely, we suppose that H^\star is a Hopf algebra dually paired with H and we use the generalised quantum double[2] in the infinite-dimensional or degenerately paired case. We also assume that the action of H on B is given by evaluation against a coaction of H^\star . The paper[4] gives the precise formula for $D(B \bowtie H)$ where B is left-covariant under H . In our case, B^\star is right-covariant under H and hence left-covariant under H^{op} , and we need $D(B^\star \bowtie H^{\text{op}})$. This contains $B^{\star\text{op}}, H, H^\star, B$ as subalgebras, with the cross relations and coproduct[4]

$$\begin{aligned} ah &= h_{(2)} a_{(2)} \langle h_{(3)}, a_{(1)} \rangle \langle h_{(1)}, S^{-1} a_{(3)} \rangle, \quad ch = h_{(2)}(c \triangleleft h_{(1)}), \quad ba = a_{(2)}(b \triangleleft \mathcal{R}^{(2)}) \langle \mathcal{R}^{(1)}, a_{(1)} \rangle \\ ac &= (c \triangleleft \mathcal{R}^{(2)}) a_{(1)} \langle \mathcal{R}^{(1)}, S^{-1} a_{(2)} \rangle, \quad bh = h_{(1)}(b \triangleleft h_{(2)}) \\ bc &= \mathcal{R}_1^{(1)} \underline{c}_{(2)} (\underline{b}_{(1)} \triangleleft \mathcal{R}_2^{(1)})^{(\bar{1})} (\underline{b}_{(2)} \triangleleft \mathcal{R}_3^{(1)}) \text{ev}(\underline{c}_{(1)}, (\underline{b}_{(1)} \triangleleft \mathcal{R}_2^{(1)})^{(\bar{2})}) \text{ev}(\underline{S}^{-1} \underline{c}_{(3)} \triangleleft \mathcal{R}_1^{(2)} \mathcal{R}_2^{(2)} \mathcal{R}_3^{(2)}, \underline{b}_{(3)}) \\ \Delta c &= \mathcal{R}^{(1)} \underline{c}_{(1)} \otimes \underline{c}_{(2)} \triangleleft \mathcal{R}^{(2)}, \quad \Delta b = \underline{b}_{(1)}^{(\bar{2})} \otimes \underline{b}_{(1)}^{(\bar{1})} \underline{b}_{(2)} \end{aligned} \quad (7)$$

for all $c \in B^\star$, $b \in B$, $h \in H$ and $a \in H^\star$, and the usual coproducts of $H, H^{\star\text{cop}}$. Here the quantum double of H in the form $H \bowtie H^{\star\text{cop}}$ appears as a sub-Hopf algebra. In the case of strict duality we have Drinfeld's quasitriangular structure as

$$\mathcal{R}_D = (\underline{S} f^a \triangleleft (S \mathcal{R}^{(2)}) u f^\alpha \otimes \mathcal{R}^{(1)} f^\beta \otimes 1 \otimes 1) \otimes (1 \otimes 1 \otimes e_\alpha e_\beta \otimes e_a) \quad (8)$$

where $\{e_a\}$ is a basis of B with dual $\{f^a\}$ and $\{e_\alpha\}$ a basis of H^\star with dual $\{f^\alpha\}$.

Theorem 2.2 [4] *There is a Hopf algebra surjection $\pi : D(B^\star \bowtie H^{\text{op}}) \rightarrow B^{\star \text{op}} \bowtie H \ltimes B$ with*

$$\pi(c) = c, \quad \pi(h) = h, \quad \pi(a) = \mathcal{R}^{(2)}\langle a, \mathcal{R}^{(1)} \rangle, \quad \pi(b) = b, \quad \forall c \in B^\star, \quad h \in H, \quad a \in H^\star, \quad b \in B.$$

In the case of strict duality this is a surjection of quasitriangular Hopf algebras.

Finally, it is known[9] that bosonisations can be viewed as examples of a more general biproduct construction (they are not the same and did not arise this way, however; see[10]); likewise, given in [4] is a more general ‘double-biproduct’ construction. The input data is a braided group B right-covariant under the quantum double $D(H)$, where H is a general Hopf algebra. It is natural to demand that H has a skew-antipode but this is not actually needed (one does not need the inverse of the braiding for the following results). Also, it is standard to write covariance under $D(H)$ slightly more generally as a (right) crossed H -modules in the standard way; see [6]. This means a compatible right action and coaction of H on B , under which it is covariant and forms a braided group. We let B^\odot be a braided group which is *left*-covariant under $D(H)$ (a left crossed H -module) and ‘skew-dual’ to B in the sense[4]

$$\begin{aligned} b^{(\bar{1})} \triangleleft c^{(\bar{1})} \otimes b^{(\bar{2})} \triangleright c^{(\bar{2})} &= b \otimes c, \quad \sigma(h \triangleright c, b) = \sigma(c, b \triangleleft h) \\ \sigma(c, ab) &= \sigma(c_{(2)}, a \triangleleft b^{(\bar{2})}) \sigma(c_{(1)}, b^{(\bar{1})}), \quad \sigma(cd, b) = \sigma(c, b_{(1)}) \sigma(d, b_{(2)}), \quad \sigma(\underline{S}c, b) = \sigma(c, \underline{S}^{-1}b) \end{aligned} \quad (9)$$

for all $h \in H$, $a, b \in B$ and $c, d \in B^\odot$, for some linear map $\sigma : B^\odot \otimes B \rightarrow \mathbb{C}$.

Theorem 2.3 [4] *For B^\odot, H, B as described, there is a unique Hopf algebra $B^\odot \bowtie H \ltimes B$ built on $B^\odot \otimes H \otimes B$, the double biproduct, containing B^\odot, H, B as subalgebras, with the cross relations and coproduct*

$$\begin{aligned} bh &= h_{(1)}(b \triangleleft h_{(2)}), \quad hc = (h_{(1)} \triangleright c)h_{(2)}, \quad bc = b_{(1)}^{(\bar{2})}c_{(2)}b_{(2)}c_{(3)}^{(\bar{1})}\sigma(c_{(1)}, b_{(1)}^{(\bar{1})})\sigma(\underline{S}c_{(3)}^{(\bar{2})}, b_{(3)}) \\ \Delta b &= b_{(1)}^{(\bar{1})} \otimes b_{(1)}^{(\bar{2})}b_{(2)}, \quad \Delta c = c_{(1)}c_{(2)}^{(\bar{1})} \otimes c_{(2)}^{(\bar{2})} \end{aligned}$$

for $b \in B$, $c \in B^\odot$ and $h \in H$, and the usual coproduct of H .

This reduces to Theorem 2.1 in the case where H is quasitriangular; and reduces to a third construction[4] when H is dual-triangular.

3 A Fermionic Cousin of $U_q(su_3)$

In this section we give a non-standard example of the double-bosonisation construction. That is, we use the R-matrix version of the double-bosonisation theorem in [4] and compute it for a non-standard R-matrix. This R-matrix double-bosonisation is built from a quantum group H with generators $\mathbf{m}^\pm = \{m^{\pm i}_j\}$, ς obeying

$$\begin{aligned} R\mathbf{m}_1^\pm \mathbf{m}_2^\pm &= \mathbf{m}_2^\pm \mathbf{m}_1^\pm R, \quad R\mathbf{m}_1^+ \mathbf{m}_2^- = \mathbf{m}_2^- \mathbf{m}_1^+ R, \quad \Delta \mathbf{m}^\pm = \mathbf{m}^\pm \otimes \mathbf{m}^\pm, \quad \epsilon \mathbf{m}^\pm = \text{id} \\ [\varsigma, \mathbf{m}^\pm] &= 0, \quad \Delta \varsigma = \varsigma \otimes \varsigma, \quad \epsilon \varsigma = 1 \end{aligned} \quad (10)$$

and braided groups B, B^\star with generators $\mathbf{e} = \{e^i\}$ and $\mathbf{f} = \{f_i\}$ respectively, obeying

$$\begin{aligned} \mathbf{e}_2 \mathbf{e}_1 &= R' \mathbf{e}_1 \mathbf{e}_2, \quad \Psi(\mathbf{e}_2 \otimes \mathbf{e}_1) = R \mathbf{e}_1 \otimes \mathbf{e}_2, \quad \underline{\Delta} \mathbf{e} = \mathbf{e} \otimes 1 + 1 \otimes \mathbf{e}, \\ \mathbf{f}_2 \mathbf{f}_1 &= \mathbf{f}_1 \mathbf{f}_2 R', \quad \Psi(\mathbf{f}_2 \otimes \mathbf{f}_1) = \mathbf{f}_1 \otimes \mathbf{f}_2 R, \quad \underline{\Delta} \mathbf{f} = \mathbf{f} \otimes 1 + 1 \otimes \mathbf{f} \end{aligned} \quad (11)$$

This is a right handed setting of the author's R-matrix braided group theory; see [6, Chapter 10] for more details and history. There are further relations beyond (10) to form a Hopf algebra, and the choice of an associated *quantum group normalisation constant* λ such that this is quasitriangular. The double-bosonisation in this case has cross relations and coproduct[4]

$$\begin{aligned} \mathbf{e}_2 \mathbf{m}_1^+ &= \lambda R \mathbf{m}_1^+ \mathbf{e}_2, \quad \mathbf{m}_2^- \mathbf{e}_1 = \lambda R \mathbf{e}_1 \mathbf{m}_2^-, \quad \mathbf{m}_1^+ \mathbf{f}_2 = \mathbf{f}_2 \mathbf{m}_1^+ \lambda R, \quad \mathbf{f}_1 \mathbf{m}_2^- = \mathbf{m}_2^- \mathbf{f}_1 \lambda R \\ \varsigma \mathbf{f} &= \lambda \mathbf{f} \varsigma, \quad \mathbf{e} \varsigma = \lambda \varsigma \mathbf{e}, \quad [\mathbf{e}, \mathbf{f}] = \frac{\mathbf{m}^+ \varsigma^{-1} - \varsigma \mathbf{m}^-}{q - q^{-1}} \\ \Delta e^i &= e^a \otimes m^{+i}{}_a \varsigma^{-1} + 1 \otimes e^i, \quad \Delta f_i = f_i \otimes 1 + \varsigma m^{-a}{}_i \otimes f_a \end{aligned} \quad (12)$$

The factor $q - q^{-1}$ is an arbitrary choice of normalisation for the e^i , chosen for conventional purposes with respect to the standard examples. The normalisation is such that the duality pairing of B^*, B is $\text{ev}(f_i, e^j) = \delta_i^j (q - q^{-1})^{-1}$.

For our example, we take $U_q(su_2)$ in more or less the standard form generated by $q^{\pm \frac{H}{2}}, X_{\pm}$ but with the opposite coproduct, and we denote by $U_q(\widetilde{su_2})$ its central extension by ς . To present our answer in a more familiar form, we also adjoin the square roots $q^{\pm \frac{H}{4}}$ and $\varsigma^{\pm \frac{1}{2}}$. We use the R-matrix form above with ansatz[3]

$$\mathbf{m}^+ = \begin{pmatrix} q^{-\frac{H}{2}} & q^{-\frac{1}{2}}(q - q^{-1})X_+ \\ 0 & q^{\frac{H}{2}} \end{pmatrix}, \quad \mathbf{m}^- = \begin{pmatrix} q^{\frac{H}{2}} & 0 \\ -q^{\frac{1}{2}}(q - q^{-1})X_- & q^{-\frac{H}{2}} \end{pmatrix} \quad (13)$$

(the ansatz embodies the additional relations mentioned above) and

$$R = - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 1 - q^{-2} & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R' = q^2 R, \quad (14)$$

which yields the fermionic quantum planes $\mathbb{C}_q^{0|2}$ and $\mathbb{C}_q^{0|2 \star \text{op}} = \mathbb{C}_q^{0|2}$ as given by

$$\eta_+ e = -q^{-1} e \eta_+, \quad e^2 = \eta_+^2 = 0, \quad \eta_- f = -q^{-1} f \eta_-, \quad f^2 = \eta_-^2 = 0. \quad (15)$$

We denote the braided (co)vector components here as $\mathbf{e} = \begin{pmatrix} e \\ \eta_+ \end{pmatrix}$ and $\mathbf{f} = (f, \eta_-)$.

Proposition 3.1 *The double-bosonisation $\mathbb{C}_q^{0|2 \star \text{op}} \bowtie U_q(\widetilde{su_2}) \bowtie \mathbb{C}_q^{0|2}$ consists of $U_q(su_2)$ with the additional generators $q^{\pm \frac{H}{2}}, \theta_{\pm}$ and the relations and coproduct*

$$\begin{aligned} [q^{\frac{H}{2}}, q^{\frac{H}{2}}] &= 0, \quad q^{\frac{H}{2}} \theta_{\pm} = q^{\mp \frac{1}{2}} \theta_{\pm} q^{\frac{H}{2}}, \quad q^{\frac{H}{2}} X_{\pm} = q^{\mp \frac{1}{2}} X_{\pm} q^{\frac{H}{2}}, \quad q^{\frac{H}{2}} \theta_{\pm} = \pm \iota \theta_{\pm} q^{\frac{H}{2}} \\ [\theta_+, X_-] &= 0, \quad [X_+, \theta_-] = 0, \quad [\theta_+, \theta_-] = \frac{q^{\frac{H}{2}} - q^{-\frac{H}{2}}}{q - q^{-1}}, \quad \theta_{\pm}^2 = 0, \quad [[X_{\pm}, \theta_{\pm}]_q, X_{\pm}]_q = 0 \\ \Delta q^{\frac{H}{2}} &= q^{\frac{H}{2}} \otimes q^{\frac{H}{2}}, \quad \Delta \theta_{\pm} \otimes q^{-\frac{H}{2}} + q^{\frac{H}{2}} \otimes \theta_{\pm} \end{aligned}$$

where $[x, y]_q = xy - qyx$ and $\iota = \sqrt{-1}$. We assume $q^2 \neq \pm 1$.

Proof It is instructive to see exactly how this arises. Firstly, the quantum group normalisation constant for (14) is $\lambda = -q^{\frac{1}{2}}$ (to bring λR into the quantum group $U_q(su_2)$ normalisation). Next, we compute the \mathbf{em}^+ relations from (12), yielding

$$eq^{-\frac{H}{2}} = q^{\frac{1}{2}} q^{-\frac{H}{2}} e, \quad eX_+ = q^{\frac{1}{2}} X_+ e, \quad \eta_+ q^{\frac{H}{2}} = q^{\frac{1}{2}} q^{\frac{H}{2}} \eta_+, \quad -q^{-\frac{1}{2}} X_+ \eta_+ + \eta_+ X_+ = q^{\frac{H}{2}} e. \quad (16)$$

The last of these says that e need not be included among the generators since it is obtained from η_+, X_+ by a q -commutator. The first relation is then implied, while the second relation remains as a 'q-Serre' relation. Similarly, the \mathbf{em}^- relations yield

$$X_- \eta_+ = q^{\frac{1}{2}} \eta_+ X_-, \quad q^{\frac{1}{2}} e X_- - q X_- e = \eta_+ q^{\frac{H}{2}}. \quad (17)$$

The computation for $\mathbf{m}^\pm \mathbf{f}$ is similar, with f generated by q -commutator of X_-, η_- . From the value of λ we obtain the cross relations

$$e\varsigma = -q^{\frac{1}{2}}\varsigma e, \quad \varsigma f = -q^{\frac{1}{2}}f\varsigma, \quad \eta_\pm \varsigma = -q^{\pm\frac{1}{2}}\varsigma \eta_\pm \quad (18)$$

and from the form of \mathbf{m}^\pm we obtain

$$[e, f] = \frac{q^{-\frac{H}{2}}\varsigma^{-1} - \varsigma q^{\frac{H}{2}}}{q - q^{-1}}, \quad [\eta_+, \eta_-] = \frac{q^{\frac{H}{2}}\varsigma^{-1} - \varsigma q^{-\frac{H}{2}}}{q - q^{-1}}, \quad [e, \eta_-] = q^{-\frac{1}{2}}X_+\varsigma^{-1}, \quad [\eta_+, f] = q^{\frac{1}{2}}\varsigma X_- \quad (19)$$

We also obtain the coproducts

$$\Delta\eta_+ = \eta_+ \otimes q^{\frac{H}{2}}\varsigma^{-1} + 1 \otimes \eta_+, \quad \Delta\eta_- = \eta_- \otimes 1 + \varsigma q^{-\frac{H}{2}} \otimes \eta_- \quad (20)$$

and more complicated coproducts for e, f . The relations involving η_\pm , along with those of $U_q(su_2)$ and $q^h = \varsigma q^{-\frac{H}{2}}$ provide our new quantum group, while those with e, f are redundant (they provide a useful check). The result is a quantum group with relations $\{q^h, \eta_\pm\} = 0$, etc. and coproduct $\Delta\eta_+ = \eta_+ \otimes q^{-h} + 1 \otimes \eta_+$ etc.

Finally, if $q^{\pm\frac{H}{4}}, \varsigma^{\pm\frac{1}{2}}$ (and $\sqrt{-1}$) are also adjoined, we can make a change of variables to

$$\theta_+ = -\eta_+ q^{\frac{h}{2}}, \quad \theta_- = q^{-\frac{h}{2}}\eta_- \quad (21)$$

This gives the form with a more familiar symmetrical coproduct as actually stated. Note that the other q -Serre relation corresponding to $\eta_+e = -q^{-1}e\eta_+$ and the remaining relation $e^2 = 0$ are implied by the relations shown, assuming $q^2 \neq -1$. Similarly for the η_-, f sector. \square

Double-bosonisation not only gives us a quantum group but constructs an ‘inductive basis’ and relations associated with it (such as the second of (17), which are usually found using the quantum Weyl group.) The natural basis is

$$f^{\epsilon_1}\eta_-^{\epsilon_2}\varsigma^m U_q(su_2)\eta_+^{\epsilon_3}e^{\epsilon_4}, \quad \epsilon_i \in \{0, 1\}, \quad m \in \mathbb{Z}, \quad (22)$$

given a basis of $U_q(su_2)$. Similarly in the θ_\pm version.

Also, the fermionic quantum plane algebra here is finite-dimensional and we have a strict duality. Taking basis $\{1, e, \eta_+, \eta_+e\}$ of B , the dual basis is

$$\{1, (q - q^{-1})f, (q - q^{-1})\eta_-, -q(q - q^{-1})^2 f\eta_-\} \quad (23)$$

where the product is written in B^{op} . Only the coefficient of $f\eta_-$ requires some work here. It is computed from

$$\text{ev}(f \cdot_{\text{op}} \eta_-, \eta_+e) = \text{ev}(f, (\eta_+e)_{(1)})\text{ev}(\eta_-, (\eta_+e)_{(2)}) \quad (24)$$

$$\underline{\Delta}(\eta_+e) = \eta_+e \otimes 1 + 1 \otimes \eta_+e + \eta_+ \otimes e + \Psi(\eta_+ \otimes e), \quad \Psi(\eta_+ \otimes e) = -q^{-1}e \otimes \eta_+ - (1 - q^{-2})\eta_+ \otimes e$$

using (11). Then (5) gives the quasitriangular structure on the double-bosonisation as

$$\mathcal{R} = \exp_{21}^{-1} \lambda^{-\xi \otimes \xi} \mathcal{R}_{su_2}, \quad \exp_{21}^{-1} = 1 \otimes 1 - (q - q^{-1})f \otimes e - q^{-1}(q - q^{-1})^2 f\eta_- \otimes \eta_+e \quad (25)$$

where we suppose that $\varsigma = \lambda^\xi$ so that $\lambda^{-\xi \otimes \xi} \mathcal{R}_{su_2}$ is the quasitriangular structure of $U_q(\widetilde{su_2})$ (under which B, B^* are covariant). We used braided-antimultiplicativity of the braided antipode to compute $\underline{S}(\eta_+e) = \cdot\Psi((- \eta_+) \otimes (-e)) = q^{-2}\eta_+e$. Unlike the similar ‘induction formula’ for \mathcal{R}_{su_3} in [4], the braided plane wave exp is a polynomial in our q -fermionic generators. On the other hand, the dilaton contribution $\lambda^{-\xi \otimes \xi}$ to the Gaussian part is more formal. It can be treated either by adjoining ξ to the algebra formally, or by working over \mathbb{C} with operator interpretations of all equations. Thus, formally,

$$\xi = \frac{2h + H}{2i\pi(\ln q)^{-1} + 1}, \quad \lambda^{-\xi \otimes \xi} = q^{-\frac{(2h+H) \otimes (2h+H)}{4i\pi(\ln q)^{-1} + 2}} \quad (26)$$

This is the ‘fermionic extension’ of $U_q(su_2)$ which is the cousin of $U_q(su_3)$ as the ‘bosonic extension’. In the same way, all the $U_q(su_n)$ have a standard extension which yields $U_q(su_{n+1})$ (in some suitable conventions) as in (1), and a corresponding fermionic extension. These are the two most natural possibilities to go from the node $U_q(su_n)$, in view of the fact that the su_n R-matrix is Hecke, i.e. has two skew-eigenvalues. We have the same two choices for extensions of the quantum line $U_q(1)$. The bosonic extension is $U_q(su_2)$ and the fermionic one is essentially the non-standard quantum group studied (with quasitriangular structure) in [11].

4 Anyonic and C-Statistical Doubles

In this section we let B be a $\mathbb{Z}/n\mathbb{Z}$ -graded or ‘anyonic’ braided group [12] and compute its double-bosonisation. By definition, an anyonic braided group means a braided group which is covariant under the anyon-generating quantum group \mathbb{Z}'_n introduced in [12]. This has a single generator g and

$$g^n = 1, \quad \Delta g = g \otimes g, \quad \mathcal{R}_g = n^{-1} \sum_{a,b=0}^{n-1} q^{-ab} g^a \otimes g^b, \quad (27)$$

where $q^n = 1$. A covariant action means that B is $\mathbb{Z}/n\mathbb{Z}$ -graded, and the corresponding (say, right) action is $b \triangleleft g = bq^{|b|}$ for b homogeneous of degree $|b|$. The induced braiding is $\Psi(b \otimes c) = c \otimes bq^{|c||b|}$ for homogeneous $b, c \in B$, and B is required to be a braided group with this braiding. We also need another braided group B^* dual to B in the sense of a map ev relating the (opposite) product of one to the coproduct of the other, and covariant in the sense $\text{ev}(c, b) = 0$ unless $|c| + |b| = 0$ for all homogeneous $c \in B^*$ and $b \in B$. Putting these formulae into Theorem 2.1, we obtain at once:

Proposition 4.1 *For B anyonic with dual B^* , the double-bosonisation $B^{\star\text{op}} \rtimes \mathbb{Z}'_n \ltimes B$ is a quantum group containing $B^{\star\text{op}}, \mathbb{Z}'_n, B$ with cross relations and coproduct*

$$\begin{aligned} bg &= gbq^{|b|}, \quad cg = gcq^{|c|}, \quad b_{(1)} g^{|c_{(2)}|} c_{(1)} \text{ev}(c_{(2)}, b_{(2)}) = \text{ev}(c_{(1)}, b_{(1)}) c_{(2)} g^{|b_{(1)}|} b_{(2)} \\ \Delta b &= b_{(1)} \otimes g^{|b_{(1)}|} b_{(2)}, \quad \Delta c = g^{|c_{(2)}|} c_{(1)} \otimes c_{(2)} \end{aligned}$$

for all $b \in B, c \in B^*$.

These computations are similar to the derivation of the formulae for single bosonisation in [6]. In the case of strict duality, we also have a quasitriangular structure $\mathcal{R} = \exp_{21}^{-1} \mathcal{R}_g$ from (5). On the other hand, when $n = 2$ any anyonic braided group means a super-quantum group. In this case, we find that we can recognise the double-bosonisation as the usual bosonisation of a suitable super-double. To this end, we first define the anyonic-opposite product on B^* , which algebra we denote $B^{\star\text{op}}$, by:

$$c \cdot_{\text{op}} d = dcq^{-|c||d|}. \quad (28)$$

Proposition 4.2 *When $n = 2$ and $q = -1$, we have $B^{\star\text{op}} \rtimes \mathbb{Z}'_n \ltimes B \cong \mathbb{Z}'_n \ltimes D(B)$, where $D(B)$ is generated by $B^{\star\text{op}}, B$ as subalgebras, with the cross relations*

$$b_{(1)} c_{(1)} \text{ev}(c_{(2)}, b_{(2)}) (-1)^{|b_{(1)}||c|} = (-1)^{|c_{(1)}||c_{(2)}|} \text{ev}(c_{(1)}, b_{(1)}) c_{(2)} b_{(2)}$$

for all $b \in B, c \in B^*$. The coproducts are those of B^*, B .

Proof The novel part here is not the definition of $D(B)$, which is analogous to Drinfeld's quantum double (up to signs), but the isomorphism $\theta : \mathbb{Z}'_n \bowtie D(B) \rightarrow B^{\star\text{op}} \bowtie \mathbb{Z}'_n \bowtie B$, which we define as

$$\theta(g) = g, \quad \theta(b) = b, \quad \theta(c) = g^{|c|}c, \quad \forall b \in B, \quad c \in B^{\star\text{op}}$$

This is clearly an isomorphism when restricted to the subalgebras $B^{\star\text{op}}, \mathbb{Z}'_n, B$, and respects the $g - b$ and $g - c$ cross relations in $\mathbb{Z}'_n \bowtie D(B)$ since these have the same form as in Proposition 4.1. Moreover, applying θ to the left hand side of the $D(B)$ relation gives

$$\begin{aligned} (-1)^{|b_{(1)}||c|} \underline{b_{(1)}} g^{|c_{(1)}|} \underline{c_{(1)}} \text{ev}(c_{(2)}, b_{(2)}) &= g^{|c|} \underline{b_{(1)}} g^{|c_{(2)}|} \underline{c_{(1)}} \text{ev}(c_{(2)}, b_{(2)}) = g^{|c|} \text{ev}(c_{(1)}, b_{(1)}) \underline{c_{(2)}} g^{|b_{(1)}|} \underline{b_{(2)}} \\ &= g^{|c|} \text{ev}(c_{(1)}, b_{(1)}) \underline{c_{(2)}} g^{|c_{(1)}|} \underline{b_{(2)}} = (-1)^{|c_{(1)}||c_{(2)}|} \text{ev}(c_{(1)}, b_{(1)}) g^{|c_{(2)}|} \underline{c_{(2)}} \underline{b_{(2)}}, \end{aligned}$$

which is θ applied to the right hand side. The coproducts also map over: the coproduct of $\mathbb{Z}'_n \bowtie D(B)$ is the same as in Proposition 4.1 on B, \mathbb{Z}'_n , and is $\Delta c = c_{(1)} \otimes g^{|c_{(1)}|} c_{(2)}$ on $B^{\star\text{op}}$. We have $(\theta \otimes \theta) \circ \Delta c = g^{|c_{(1)}|} c_{(1)} \otimes g^{|c_{(2)}|} c_{(2)} = g^{|c|} g^{|c_{(2)}|} c_{(1)} \otimes g^{|c_{(2)}|} c_{(2)} = \Delta \circ \theta(c)$. \square

When $n > 2$, we still obtain an ordinary quantum group by Proposition 4.1 but it is not (as far as I know) the single bosonisation of anything. This is because the braided quantum double construction does not really work (it gets tangled up) and, indeed, the double-bosonisation is probably the closest we can come to it. This was one of the motivations in [4].

Finally, we are not limited to one-dimensional gradings. In [13] we introduced quasitriangular Hopf algebras $U_q(\beta)$ (say) associated to any bilinear form β_{ij} . The generators are commuting variables $\{\xi_i\}$ with structure

$$\Delta \xi_i = \xi_i \otimes 1 + 1 \otimes \xi_i, \quad \mathcal{R}_\xi = q \sum \beta_{ij} \xi_i \otimes \xi_j, \quad (29)$$

where q is general. The action on a \mathbb{Z}^n -graded algebra is $b \triangleleft \xi_i = b |b|_i$ where $|b|_i$ is the i -component of the degree. Then $\Psi(b \otimes c) = c \otimes b q^{\sum \beta_{ij} |c|_i |b|_j}$ is the braiding. Braided groups in this setting are called \mathbb{C} -statistical[13].

An example of B, B^* in this setting is provided[13] by the free algebras $B = \mathbb{C}\langle e^i \rangle$, $B^* = \mathbb{C}\langle f_i \rangle$, with grading and resulting braiding and pairing

$$|e^i|_j = \delta^i_j, \quad \Psi(e^i \otimes e^j) = e^j \otimes e^i q^{\beta_{ji}}, \quad |f_i|_j = -\delta_{ij}, \quad \Psi(f_i \otimes f_j) = f_j \otimes f_i q^{\beta_{ji}}, \quad \text{ev}(f_i, e^j) = \delta_i^j (q^i - q_i^{-1})^{-1}. \quad (30)$$

This is simply the R-matrix setting (11) with $R^i_j{}^k{}_l = \delta^i_j \delta^k_l q^{\beta_{ki}}$ and $R' = P$, the permutation matrix. (More generally, one can have *free braided groups* associated to any R and $R' = P$, as studied extensively in [14].) The additional parameters denoted $q^i - q_i^{-1}$ reflect some freedom in the normalisation of the e^i . From Theorem 2.1, we have immediately:

Proposition 4.3 *The double-bosonisation $\mathbb{C}\langle \mathbf{f} \rangle \bowtie U_q(\beta) \bowtie \mathbb{C}\langle \mathbf{e} \rangle$ has mutually non-commuting $\{e^i\}$, mutually non-commuting $\{f_j\}$, mutually commuting $\{\xi_i\}$ and cross relations and coproduct obeying*

$$\begin{aligned} e^i q^{H^j} &= q^{H^j} e^i q^{\beta_{ji}}, \quad f_i q^{H_j} = q^{H_j} f_i q^{-\beta_{ij}}, \quad [e^i, f_j] = \delta^i_j \frac{q^{H^i} - q^{-H_i}}{q_i - q_i^{-1}} \\ \Delta e^i &= e^i \otimes q^{H^i} + 1 \otimes e^i, \quad \Delta f_i = f_i \otimes 1 + q^{-H_i} \otimes f_i; \quad H^i \equiv \sum \beta_{ij} \xi_j, \quad H_i \equiv \sum \xi_j \beta_{ji}. \end{aligned}$$

The computation is similar to the single bosonisation of (30) in [13]. We assume for simplicity here that β is invertible – otherwise we have to work with the ξ variables, with relations $[e^i, \xi_j] = \delta^i_j e^i$ and $[f_i, \xi_j] = -\delta_{ij} f_i$. Also, we have presented the H^i, H_i symmetrically, but either set will suffice.

The pairing ev between the free braided groups (30) is typically degenerate, but we can always quotient out by the kernels of the pairing on each side. When β_{ij} is the symmetric bilinear form associated to a

Cartan matrix and $q^i = q_i = q^{\frac{\beta_{ii}}{2}}$, Lusztig[15] has effectively computed the kernels and found that they are generated by the q -Serre relations. Hence, quotienting by these gives \mathbb{C} -statistical braided groups $U_q(n_{\pm})$ and

$$U_q(n_-) \bowtie U_q(\beta) \bowtie U_q(n_+) = U_q(g) \quad (31)$$

recovers Lusztig's construction of $U_q(g)$ in our braided approach as a double-bosonisation. We have glossed over a lot of technicalities here (we need to work with q^{H_i} as generators and avoid the power series inherent in \mathcal{R}_{ξ} , by working with weak quasitriangular structure maps); see [4] for the full details.

On the other hand, as explained in [4], we are not limited to β_{ij} associated to a Cartan matrix. When we take some other bilinear form (typically integer-valued, for an algebraic answer) we have some other kernel of ev in (30). The extension of ev to products is via braided differentiation, so the kernels have a 'braided geometrical' interpretation[14][6]. Quotienting out by the kernels, we obtain non-degenerately paired braided groups B, B^* . Their double-bosonisation gives new quantum groups complete with quasitriangular structure $\exp_{21}^{-1} \mathcal{R}_{\xi}$, where $\exp \in B \otimes B^*$ is the braided exponential or coevaluation for the pairing ev , generally existing as a formal power series. Some of these quantum groups will be computed elsewhere.

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